

# LINEAR FLOWS ON $\kappa$ -SOLENOIDS

ALEX CLARK

ABSTRACT. Linear flows on inverse limits of tori are defined and it is shown that two linear flows on an inverse limit of tori are equivalent if and only if there is an automorphism of the inverse limit generating the equivalence.

## 1. INTRODUCTION

We define the families of linear flows on the inverse limits of finite-dimensional tori  $\mathbf{T}^n$  ( $n$  fixed) with epimorphic bonding maps and on a special class of inverse limits of  $\mathbf{T}^\infty$  and prove that two flows from such a family are topologically equivalent if and only if there is an automorphism generating the equivalence. This result generalizes the well known classification of the linear flows on  $\mathbf{T}^2$  (see [I], pp. 36-38). While there does not seem to be a proof in the literature of this result for the general linear flow on  $\mathbf{T}^\kappa$  for  $\kappa > 2$ , some related questions are addressed in ([KH], 2.3). This result reduces the problem of the classification of these linear flows to the classification of the automorphisms of the corresponding inverse limit. We give a characterization of the automorphisms on the finite product of one-dimensional solenoids and we work out the 2-dimensional case in detail, thereby classifying the linear flows on such products. We also find a condition on the character group of a finite-dimensional inverse limit as above that determines when the inverse limit is isomorphic with a product of one-dimensional solenoids.

It can be shown that the subgroup of the reals generated by the Bohr-Fourier exponents of an almost periodic orbit of a flow in a complete metric space determines the equivalence class of the flow obtained by the restriction of the original flow to the compact minimal set which the closure of the image of the orbit forms in the sense that any two such orbits with the same associated group determine equivalent flows (see, for example, [LZ], 3§2). This, together with a straightforward application of Pontryagin duality, can be used to demonstrate that a flow in a complete metric space restricted to the closure of the image of an almost periodic orbit is equivalent to an irrational linear flow as defined here. Any two (topologically) equivalent almost periodic flows are equivalent to members of the same family of irrational linear flows, and so our results serve as a program for the classification of almost periodic flows in complete metric spaces. It then follows from a theorem of Nemytskii ([NS]; V, 8.16) that any metric compact connected abelian group is isomorphic with an inverse limit of the type treated here (a  $\kappa$ -solenoid) (see also [Pont], Thm 68).<sup>1</sup>

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<sup>1</sup> A more detailed exposition of these matters appears in the dissertation of the author.

2.  $\kappa$ -SOLENOIDS

**2.1.  $n$ -Solenoids.**  $S^1 \stackrel{def}{=} \mathbf{T}^1 \stackrel{def}{=} \mathbb{R}/\mathbb{Z}$ , with group operation “+” inherited from the covering homomorphism  $p^1 : \mathbb{R} \rightarrow S^1$ ;  $x \mapsto x \pmod{1}$ . Unless otherwise stated,  $n$  denotes a member of the set of natural numbers  $\mathbb{N} = \{1, 2, \dots\}$  and  $\infty$  denotes the countably infinite cardinal. The  $n$ -torus is denoted  $\mathbf{T}^n \stackrel{def}{=} \prod_{i=1}^n S^1$  and  $\mathbf{T}^\infty$  is defined to be the space  $\prod_{i=1}^\infty S^1$  and we let  $\mathbf{x} = \langle x_1, \dots, x_i, \dots \rangle$  denote a point of  $\mathbf{T}^\kappa$  for  $\kappa \in \mathbb{N}$  or  $\kappa = \infty$ , and we give  $\mathbf{T}^\kappa$  the metric  $d_\kappa$ ;

$$d_\kappa(\mathbf{x}, \mathbf{y}) \stackrel{def}{=} \sum_{i=1}^\kappa \frac{1}{2^i} |u_i - v_i|,$$

where  $u_i, v_i \in \mathbb{R}$  are representatives of the classes of  $x_i, y_i$  chosen so that  $|u_i - v_i| \leq \frac{1}{2}$ . An inverse limit of  $\mathbf{T}^\kappa$  ( $\kappa \in \mathbb{N}$  or  $\kappa = \infty$ , fixed) with epimorphic bonding maps has the group structure inherited from the Cartesian product  $\prod_{i=1}^\infty \mathbf{T}^\kappa$ . For a fixed  $\kappa$ ,  $\prod_{i=1}^\infty \mathbf{T}^\kappa$  and its subspaces are given the metric  $d_\kappa^\infty$ ;

$$d_\kappa^\infty(\langle \mathbf{x}^j \rangle_{j=1}^\infty, \langle \mathbf{y}^j \rangle_{j=1}^\infty) \stackrel{def}{=} \sum_{j=1}^\infty \frac{1}{2^j} d_\kappa(\mathbf{x}^j, \mathbf{y}^j).$$

We assume throughout that all homomorphisms (automorphisms, etc.) between topological groups are continuous.

**Definition 2.1.** For  $\kappa \in \mathbb{N} \cup \{\infty\}$ ,  $p^\kappa : \mathbb{R}^\kappa \rightarrow \mathbf{T}^\kappa$  is the homomorphism  $\mathbf{t} = (t_1, \dots, t_i, \dots) \mapsto \langle p^1(t_1), \dots, p^1(t_i), \dots \rangle$ .

Notice that if  $f : \mathbf{T}^n \rightarrow \mathbf{T}^n$  is a homomorphism there is a unique homomorphism represented by a matrix with integer entries  $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying  $f \circ p^n(\mathbf{t}) = p^n \circ M(\mathbf{t})$ , and when  $f$  is an epimorphism  $\det M \in \mathbb{Z} - \{0\}$ . And if  $f : \mathbf{T}^n \rightarrow \mathbf{T}^n$  is represented by the  $n \times n$  integer matrix  $M$  with  $\det M \neq 0$ , there are two  $n \times n$  integer matrices  $P$  and  $Q$  which have inverses with integer entries and which satisfy  $M = P\Delta Q$ , where  $\Delta$  is a diagonal matrix with integer entries. Then with  $k = |\det M| = |\det \Delta|$ ,  $f$  is a  $k$ -to-one covering map since  $\Delta$  represents such a map.

**Definition 2.2.** For a fixed  $n$  and a sequence  $\overline{M} = (M_1, M_2, \dots)$  of  $n \times n$  matrices  $M_i$  with integer entries and non-zero determinants, we define the topological group  $\sum_{\overline{M}}$  with identity  $e_{\overline{M}}$  to be the inverse limit of the inverse sequence  $\{\mathbf{X}_j, f_j^i\}$ , where  $\mathbf{X}_j = \mathbf{T}^n$  for all  $j \in \mathbb{N}$  and  $f_j^{j+1}$  is the topological epimorphism represented by the matrix  $M_j$ ;  $f_j^{j+1} \circ p^n = p^n \circ M_j$ .

$$\sum_{\overline{M}} \stackrel{def}{=} \lim_{\leftarrow} \{\mathbf{X}_j, f_j^i\} \subset \prod_{j=1}^\infty \mathbf{T}^n,$$

and we define such an inverse limit  $\sum_{\overline{M}}$  to be an  $n$ -solenoid.

2.2.  $\infty$ -Solenoids.

**Definition 2.3.** If  $f : \mathbf{T}^n \rightarrow \mathbf{T}^n$ ;  $\langle x_1, \dots, x_n \rangle \mapsto \langle y_1, \dots, y_n \rangle$  is a homomorphism represented by a matrix with integer entries  $M : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ;  $(t_1, \dots, t_n) \mapsto (s_1, \dots, s_n)$ , then we define the following maps:  $f \times id : \mathbf{T}^\infty \rightarrow \mathbf{T}^\infty$ ;

$$\langle x_1, x_2, \dots \rangle \mapsto \langle y_1, \dots, y_n, x_{n+1}, x_{n+2}, \dots \rangle$$

and  $M \times id : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$  ;

$$(t_1, t_2, \dots) \mapsto (s_1, \dots, s_n, t_{n+1}, t_{n+2}, \dots).$$

And we define a map such as  $f \times id$  to be an  $n$ -map.

Notice that if  $M$  represents  $f$ , then  $M \times id$  represents  $f \times id$  in the sense that  $(f \times id) \circ p^\infty(\mathbf{t}) = p^\infty \circ (M \times id)(\mathbf{t})$ . Also, if  $f \times id$  and  $g \times id$  are both  $n$ -maps, then  $(f \times id) \circ (g \times id)$  is the  $n$ -map  $(f \circ g) \times id$ . And if  $f \times id$  is an  $n$ -map ;

$$\langle x_1, x_2, \dots \rangle \mapsto \langle y_1, \dots, y_n, x_{n+1}, x_{n+2}, \dots \rangle$$

and  $\nu > n$ , it is possible to represent  $f \times id$  as the  $\nu$ -map  $f' \times id$ , where  $f' : \mathbf{T}^\nu \rightarrow \mathbf{T}^\nu$ ;

$$\langle x_1, \dots, x_\nu \rangle \mapsto \langle y_1, \dots, y_n, x_{n+1}, \dots, x_\nu \rangle.$$

Also, if  $f \times id$  is an  $n$ -map and  $g \times id$  is an  $m$ -map, then  $(f \times id) \circ (g \times id)$  is a  $\max\{m, n\}$ -map, for we may represent both  $f \times id$  and  $g \times id$  as  $\max\{m, n\}$ -maps and then the above observation on compositions applies.

**Definition 2.4.** If for each  $i \in \mathbb{N}$   $g_i^{i+1} \times id$  is an  $n_i$ -map represented by the map  $M_i \times id : \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ , where  $M_i$  is an  $n_i \times n_i$  integer matrix with non-zero determinant, we define the topological group  $\sum_{\overline{M}}$  with identity  $e_{\overline{M}}$  to be the inverse limit of the inverse sequence  $\{G_i, f_i^j\}$ , where  $G_i = \mathbf{T}^\infty$  for all  $i \in \mathbb{N}$  and  $f_i^{i+1} = g_i^{i+1} \times id$ ;

$$\sum_{\overline{M}} \stackrel{\text{def}}{=} \varprojlim \{G_i, f_i^j\} \subset \prod_{j=1}^{\infty} \mathbf{T}^\infty,$$

and we define such an inverse limit  $\sum_{\overline{M}}$  to be an  $\infty$ -solenoid.

In the following  $\sum_{\overline{M}}$  represents a  $\kappa$ -solenoid for some  $\kappa \in \mathbb{N} \cup \{\infty\}$ .

**Definition 2.5.**  $f_i : \sum_{\overline{M}} \rightarrow \mathbf{T}^\kappa$ ;  $(\mathbf{x}^1, \mathbf{x}^2, \dots) \mapsto \mathbf{x}^i$  is projection onto the  $i^{\text{th}}$  factor.

**Definition 2.6.**  $C_{\overline{M}}$  is the path component of  $e_{\overline{M}}$  in  $\sum_{\overline{M}}$ .

**Definition 2.7.** If  $\kappa = n < \infty$ , we define  $\pi_{\overline{M}} : \mathbb{R}^n \rightarrow \sum_{\overline{M}}$  to be the homomorphism

$$\mathbf{t} = (t_1, \dots, t_n) \mapsto (p^n(\mathbf{t}), p^n \circ M_1^{-1}(\mathbf{t}), \dots, p^n \circ M_j^{-1} \circ \dots \circ M_1^{-1}(\mathbf{t}), \dots).$$

and if  $\kappa = \infty$ , we define  $\pi_{\overline{M}} : \mathbb{R}^\infty \rightarrow \sum_{\overline{M}}$  to be the homomorphism

$$\mathbf{t} \mapsto (p^\infty(\mathbf{t}), p^\infty \circ (M_1^{-1} \times id)(\mathbf{t}), \dots, p^\infty \circ (M_j^{-1} \times id) \circ \dots \circ (M_1^{-1} \times id)(\mathbf{t}), \dots),$$

Notice that  $\ker \pi_{\overline{M}} \subset \ker(f_1 \circ \pi_{\overline{M}}) = \mathbb{Z}^\kappa$ .

That  $\pi_{\overline{M}}$  maps  $\mathbb{R}^n$  onto  $C_{\overline{M}}$  follows from Theorem 5.8 in [McC], where he characterizes the path components of the inverse limit of an inverse sequence with all bonding maps regular covering maps between spaces which admit a universal covering. However, we shall demonstrate directly that  $\pi_{\overline{M}}(\mathbb{R}^n) = C_{\overline{M}}$  and generalize this result in **Corollary 2**, and to do so we prove two preliminary lemmas using the terminology and results in ([S], Chapt 2).

**Lemma 2.8.** Let  $p_1 : E \rightarrow L$  and  $p_2 : L \rightarrow B$  be maps which satisfy the following conditions: **(1)**  $p_2$  has unique path lifting and **(2)**  $p = p_2 \circ p_1$  is a fibration. Then  $p_1$  is a fibration.

**Proof:** If  $F, f'$  are maps as in the following diagram, we need to find a map  $F' : X \times [0, 1] \rightarrow E$  (represented by the diagonal arrow in the diagram) which makes the following diagram (A) commute:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f'} & E \\ \downarrow \cap & \nearrow & \downarrow p_1 \\ X \times [0, 1] & \xrightarrow{F} & L \end{array} .$$

Then for any  $x \in X$ ,  $p_1 \circ f'(x, 0) = F(x, 0)$ . Since  $p$  is a fibration, there is a map  $G : X \times [0, 1] \rightarrow E$  making the following diagram (B) commute:

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f'} & E \\ \downarrow \cap & \nearrow G & \downarrow p \\ X \times [0, 1] & \xrightarrow{p_2 \circ F} & B \end{array} .$$

Then for any  $x \in X$ ,  $p_1 \circ G(x, 0) = p_1 \circ f'(x, 0) = F(x, 0)$ ; the first equality follows from diagram (B) and the second equality from the observation after diagram (A). Now fix  $x \in X$  and define the paths  $\omega$  and  $\omega'$  in  $L$ :

$$\omega(t) \stackrel{def}{=} F(x, t) \text{ and } \omega'(t) \stackrel{def}{=} p_1 \circ G(x, t) .$$

Then we have  $\omega(0) = F(x, 0) = p_1 \circ G(x, 0) = \omega'(0)$ . It follows from diagram (B) that for all  $t \in [0, 1]$

$$p_2 \circ \omega'(t) = p_2 \circ p_1 \circ G(x, t) = p \circ G(x, t) = p_2 \circ F(x, t) = p_2 \circ \omega(t) .$$

From this and the condition that  $p_2$  has unique path lifting, it follows that for all  $t \in [0, 1]$

$$F(x, t) = \omega(t) = \omega'(t) = p_1 \circ G(x, t) .$$

Since  $x$  was any point of  $X$ , we have that  $F = p_1 \circ G$  and setting  $F' = G$  gives us the map we need to complete diagram (A), demonstrating that  $p_1$  is a fibration.  $\square$

**Lemma 2.9.** *Let  $X_\infty = \varprojlim \{f_i^j, X_i\}$  be the inverse limit of an inverse sequence for which all the bonding maps  $f_i^j$  ( $i \leq j$ ) have unique path lifting. Then the projection onto the first coordinate  $f_1 : X_\infty \rightarrow X_1$ ;  $(x_1, x_2, \dots) \mapsto x_1$  has unique path lifting.*

**Proof:** Given paths  $\omega$  and  $\omega'$  in  $X_\infty$  such that  $f_1 \circ \omega = f_1 \circ \omega'$  and  $\omega(0) = \omega'(0)$ , suppose that for some  $t \in [0, 1]$  we have  $\omega(t) \neq \omega'(t)$ . Then for some  $n \in \mathbb{N}$ ,  $\omega(t)$  and  $\omega'(t)$  disagree on the  $n^{th}$  coordinate:  $f_n \circ \omega(t) \neq f_n \circ \omega'(t)$ . But we also have that

$$f_1^n \circ f_n \circ \omega = f_1 \circ \omega = f_1 \circ \omega' = f_1^n \circ f_n \circ \omega'$$

and by hypothesis  $f_1^n$  has unique path lifting. This combined with  $f_n \circ \omega(0) = f_n \circ \omega'(0)$  implies that the paths  $f_n \circ \omega$  and  $f_n \circ \omega'$  in  $X_n$  are equal. This contradicts  $f_n \circ \omega(t) \neq f_n \circ \omega'(t)$ . Therefore, no such  $t$  can exist and  $\omega = \omega'$ .  $\square$

**Theorem 1.** *Let  $X_\infty = \varprojlim \{f_i^j, X_i\}$  be the inverse limit of an inverse sequence for which all the bonding maps  $f_i^j$  ( $i \leq j$ ) have unique path lifting and suppose that we have a map  $p : \tilde{X} \rightarrow X_\infty$  satisfying the condition that  $f_1 \circ p$  is a fibration. Then*

$p$  is a fibration, and if  $\tilde{X}$  is path connected,  $p(\tilde{X})$  is a path component of  $X_\infty$ . If each of the fibers of  $f_1 \circ p$  is totally disconnected, then  $p$  has unique path lifting.

**Proof:** Since  $f_1$  has unique path lifting by the above lemma, **Lemma 2.8** applies, implying that  $p$  is a fibration. So when  $\tilde{X}$  is path connected,  $p(\tilde{X})$  is a path component of  $X_\infty$  [S; 2.3.1].

Also, if each of the fibers of  $f_1 \circ p$  is totally disconnected, then for any  $b \in X_\infty$  we have

$$p^{-1}(b) \subset p^{-1}(f_1^{-1}(f_1(b))) = (f_1 \circ p)^{-1}(f_1(b))$$

and  $(f_1 \circ p)^{-1}(f_1(b))$  is a totally disconnected set (see [S; 2.2.5]).  $\square$

Notice that we are not requiring our spaces  $X_i$  to be groups and the above theorem could be extended to include general inverse limits [not just the inverse limits of inverse sequences].

*Corollary 2.* For  $\kappa \in \mathbb{N} \cup \{\infty\}$ , let  $\pi_{\overline{M}} : \mathbb{R}^\kappa \rightarrow \sum_{\overline{M}}$  be as in **Definition 2.7**. Then  $\pi_{\overline{M}}$  is a fibration with unique path lifting onto  $\mathbf{C}_{\overline{M}}$ .  $\square$

### 3. LINEAR FLOWS ON $\kappa$ -SOLENOIDS

**Definition 3.1.** For  $\omega = (\omega_1, \dots, \omega_j, \dots) \in \mathbb{R}^\kappa - \{\mathbf{0}\}$ , we define  $i^\omega : \mathbb{R} \rightarrow \mathbb{R}^\kappa$  by  $t \mapsto (t\omega_1, \dots, t\omega_j, \dots) \stackrel{def}{=} t\omega$ .

Notice that  $i^\omega$  topologically embeds  $\mathbb{R}$  as a subgroup of  $\mathbb{R}^\kappa$ .

**Definition 3.2.** A flow on the space  $X$  is a map  $\phi : \mathbb{R} \times X \rightarrow X$  satisfying the following conditions

1.  $\phi(0, x) = x$  for all  $x \in X$
2.  $\phi(s, \phi(t, x)) = \phi(s+t, x)$  for all  $s, t \in \mathbb{R}$ .

**Definition 3.3.** We define the family of linear flows on the  $\kappa$ -solenoid  $\sum_{\overline{M}}$

$$\mathcal{F}_{\overline{M}} = \{\Phi_{\overline{M}}^\omega \mid \omega \in \mathbb{R}^\kappa - \{\mathbf{0}\}\} \text{ to be given by}$$

$$\Phi_{\overline{M}}^\omega : \mathbb{R} \times \sum_{\overline{M}} \xrightarrow{(i^\omega, id)} \mathbb{R}^\kappa \times \sum_{\overline{M}} \xrightarrow{(\pi_{\overline{M}}, id)} \mathbf{C}_{\overline{M}} \times \sum_{\overline{M}} \xrightarrow{\pm} \sum_{\overline{M}}.$$

It follows directly that each  $\Phi_{\overline{M}}^\omega$  is indeed a flow. Notice that any time- $t$  map  $\Phi_{\overline{M}}^\omega(t, -)$  is simply translation by  $\pi_{\overline{M}}(t\omega)$ . This family of flows yields isotopies between  $id_{\sum_{\overline{M}}}$  and the translations by elements of  $\mathbf{C}_{\overline{M}}$ , and if we replace  $\mathbf{C}_{\overline{M}}$  with the path component of  $y \in \sum_{\overline{M}} - \mathbf{C}_{\overline{M}}$  and  $\pi_{\overline{M}}$  by  $y + \pi_{\overline{M}}$  in the definition of  $\Phi_{\overline{M}}^\omega$ , we obtain isotopies between the translations of elements in that path component, but these isotopies are not flows.

**Definition 3.4.** The countable set of real numbers  $\{\omega_1, \dots, \omega_i, \dots\}$  is rationally independent if :

$$[k_1\omega_{i_1} + \dots + k_s\omega_{i_s} = 0 \text{ for integers } k_1, \dots, k_s \text{ (s finite)}] \Rightarrow [k_1 = \dots = k_s = 0],$$

and in this case  $\omega = (\omega_1, \dots, \omega_i, \dots)$  and the linear flow  $\Phi_{\overline{M}}^\omega$  are irrational.

**Lemma 3.5.** *If  $\{\omega_1, \dots, \omega_n\}$  is rationally independent and  $N$  is an  $n \times n$  invertible matrix with rational entries, then  $\{\omega'_1, \dots, \omega'_n\}$  is rationally independent, where*

$$\begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_n \end{pmatrix} = N \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix}.$$

**Proof:** Suppose that for the integers  $k_1, \dots, k_n$  we have  $k_1\omega'_1 + \dots + k_n\omega'_n = 0$ . Then

$$\begin{pmatrix} k_1 & \dots & k_n \end{pmatrix} \begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_n \end{pmatrix} = \begin{pmatrix} k_1 & \dots & k_n \end{pmatrix} N \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = \begin{pmatrix} q_1 & \dots & q_n \end{pmatrix} \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_n \end{pmatrix} = 0$$

where  $q_1, \dots, q_n$  are rational numbers. The rational independence of  $\{\omega_1, \dots, \omega_n\}$  then implies that  $q_1 = \dots = q_n = 0$ . Thus,

$$\begin{pmatrix} k_1 & \dots & k_n \end{pmatrix} N = \begin{pmatrix} 0 & \dots & 0 \end{pmatrix} \Rightarrow N^T \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = 0$$

and so  $\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} \in \ker N^T = \left\{ \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$  and  $\{\omega'_1, \dots, \omega'_n\}$  is rationally independent by definition.  $\square$

**Definition 3.6.**  $\Lambda_{\overline{M}}^\omega \stackrel{\text{def}}{=} \Phi_{\overline{M}}^\omega(\mathbb{R} \times \{e_{\overline{M}}\}) = \{\pi_{\overline{M}}(t\omega) : t \in \mathbb{R}\} \subset \mathbf{C}_{\overline{M}}.$

Notice that  $\Lambda_{\overline{M}}^\omega$  is the trajectory of  $e_{\overline{M}}$  for the flow  $\Phi_{\overline{M}}^\omega$  and that  $\Lambda_{\overline{M}}^\omega$  is a subgroup of  $\Sigma_{\overline{M}}$  since it is the image of  $\mathbb{R}$  under the homomorphism  $t \mapsto \pi_{\overline{M}}(t\omega)$ .

**Lemma 3.7.** *If  $\omega$  is irrational,  $\Lambda_{\overline{M}}^\omega$  is dense in the  $\kappa$ -solenoid  $\Sigma_{\overline{M}}$ .*

**Proof:** Let  $\mathbf{x} = (\mathbf{x}^i)_{i=1}^\infty$  be any point in  $\Sigma_{\overline{M}}$  and let  $N$  be any neighborhood of  $\mathbf{x}$ . We need to show that  $N$  contains some point of  $\Lambda_{\overline{M}}^\omega$ . Since  $\mathcal{B} = \{f_i^{-1}(U) : U \text{ is open in } \mathbf{T}^\kappa, i \in \mathbb{N}\}$  is a basis for the topology of  $\Sigma_{\overline{M}}$ , there is a  $j \in \mathbb{N}$  and a neighborhood  $U$  of  $\mathbf{x}^j$  in  $\mathbf{T}^\kappa$  satisfying:  $f_j^{-1}(U) \subset N$ . Since  $U$  is a neighborhood of  $\mathbf{x}^j$  in  $\mathbf{T}^\kappa$ , there is an  $\varepsilon > 0$  such that the ball  $B$  of radius  $\varepsilon$  centered at  $\mathbf{x}^j$  in  $\mathbf{T}^\kappa$  is contained in  $U$ . We have 2 cases:  $\kappa = n < \infty$  and  $\kappa = \infty$  and we treat the second case; the first case may be proved using a simplified version of the same argument.

So we assume  $\kappa = \infty$  and seek a point  $\pi_{\overline{M}}(t\omega)$  of  $\Lambda_{\overline{M}}^\omega$  satisfying  $d_\infty(\mathbf{x}^j, f_j(\pi_{\overline{M}}(t\omega))) < \varepsilon$ . Such a point will then be contained in  $N \cap \Lambda_{\overline{M}}^\omega$  since  $f_j^{-1}(B) \subset f_j^{-1}(U) \subset N$ . First we choose  $m$  so that  $\sum_{i=m+1}^\infty \frac{1}{2^i} < \frac{\varepsilon}{2}$ . Then we represent the map  $(M_j^{-1} \times id) \circ \dots \circ (M_1^{-1} \times id)$  as a map  $M \times id$ , where  $M$  is an invertible  $k \times k$  matrix for some integer  $k \geq m$ . Then with

$$\begin{pmatrix} \omega'_1 \\ \vdots \\ \omega'_k \\ \vdots \end{pmatrix} \stackrel{\text{def}}{=} (M \times id) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \\ \vdots \end{pmatrix},$$

the set  $\{\omega'_1, \dots, \omega'_k\}$  and hence  $\{\omega'_1, \dots, \omega'_m\}$  is rationally independent by **Lemma 3.5**. Kronecker's theorem (see, e.g., [HW], Thm 444) then yields integers  $p_1, \dots, p_n$  and a real number  $t$  which satisfy the following system of inequalities:

$$\left| t\omega'_1 - p_1 - x_1^j \right| < \frac{\varepsilon}{2}, \dots, \left| t\omega'_m - p_m - x_m^j \right| < \frac{\varepsilon}{2},$$

where for  $i = 1, \dots, m$   $x_i^j$  (and hence  $p_i + x_i^j$ ) are representatives in  $\mathbb{R}$  for the coordinates of  $\mathbf{x}^j$ . Then  $d_\infty(\mathbf{x}^j, f_j(\pi_{\overline{M}}(t\omega))) < \varepsilon$  as required.  $\square$

**Definition 3.8.** The flow  $\phi : \mathbb{R} \times X \rightarrow X$  is equivalent to the flow  $\psi : \mathbb{R} \times Y \rightarrow Y$  if there is a homomorphism  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  and a homeomorphism  $h : X \rightarrow Y$  such that

$$\begin{array}{ccc} \mathbb{R} \times X & \xrightarrow{\phi} & X \\ \downarrow \alpha \times h & & \downarrow h \\ \mathbb{R} \times Y & \xrightarrow{\psi} & Y \end{array}$$

commutes and  $\alpha$  is increasing [I, pp.31-2], and we write

$$\alpha \times h : \phi \overset{\text{equiv}}{\approx} \psi.$$

(This is also sometimes referred to as  $C^0$  conjugacy).

**Lemma 3.9.** Let  $\Phi_{\overline{M}}^\omega$  be a linear flow on  $\sum_{\overline{M}}$ . If  $\tau$  is translation by  $x \in \sum_{\overline{M}}$ , then  $id_{\mathbb{R}} \times \tau : \Phi_{\overline{M}}^\omega \overset{\text{equiv}}{\approx} \Phi_{\overline{M}}^\omega$ .

**Proof:** We have

$$\begin{aligned} \tau \circ \Phi_{\overline{M}}^\omega(t, y) &= \tau(\pi_{\overline{M}}(t\omega) + y) = \pi_{\overline{M}}(t\omega) + x + y \\ &= \Phi_{\overline{M}}^\omega(t, x + y) = \Phi_{\overline{M}}^\omega[(id_{\mathbb{R}} \times \tau)(t, y)]. \end{aligned}$$

$\square$

Suppose that  $\alpha \times g : \Phi_{\overline{M}}^\omega \overset{\text{equiv}}{\approx} \Phi_{\overline{M}}^{\omega'}$ . Then with  $\tau$  defined to be translation by  $-g(e_{\overline{M}})$ , we have by the above lemma:

$$id_{\mathbb{R}} \times \tau : \Phi_{\overline{M}}^{\omega'} \overset{\text{equiv}}{\approx} \Phi_{\overline{M}}^{\omega'}$$

and so with  $h = \tau \circ g$  we obtain a homeomorphism  $h$  which fixes  $e_{\overline{M}}$  and

$$\alpha \times h = (id_{\mathbb{R}} \times \tau) \circ (\alpha \times g) : \Phi_{\overline{M}}^\omega \overset{\text{equiv}}{\approx} \Phi_{\overline{M}}^{\omega'}.$$

Thus, to determine the  $\overset{\text{equiv}}{\approx}$  classes of the family  $\mathcal{F}_{\overline{M}}$  of linear flows on  $\sum_{\overline{M}}$  we need only consider equivalences which are induced by homeomorphisms of  $\sum_{\overline{M}}$  which fix  $e_{\overline{M}}$ . We shall see that in fact we need only consider equivalences induced by automorphisms. In order to calculate the entropy of automorphisms of  $n$ -solenoids (in our terminology), Lind and Ward [LW] show that the general automorphism on an  $n$ -solenoid can be represented by a matrix in  $GL(n, \mathbb{Q})$ , as determined by the dual automorphism on the character group of the solenoid. We provide here a direct representation of automorphisms on  $\kappa$ -solenoids by automorphisms of  $\mathbb{R}^\kappa$ .

**Theorem 3.** Let  $h$  be a homomorphism from the  $\kappa$ -solenoid  $\sum_{\overline{M}}$  to the  $\kappa'$ -solenoid  $\sum_{\overline{M'}}$ . Then there is a unique homomorphism  $H : \mathbb{R}^\kappa \rightarrow \mathbb{R}^{\kappa'}$  making the following

diagram commute

$$\begin{array}{ccc} \mathbb{R}^\kappa & \xrightarrow{H} & \mathbb{R}^{\kappa'} \\ \pi_{\overline{M}} \downarrow & & \downarrow \pi_{\overline{M}'} \\ \sum_{\overline{M}} & \xrightarrow{h} & \sum_{\overline{M}'} \end{array}.$$

And the function  $f : Hom(\sum_{\overline{M}}, \sum_{\overline{M}'}) \rightarrow Hom(\mathbb{R}^\kappa, \mathbb{R}^{\kappa'})$ ;  $h \mapsto H$  is one-to-one. And if  $h$  is an automorphism,  $H$  is an automorphism.

**Proof:** Let  $h$  be a homomorphism from the  $\kappa$ -solenoid  $\sum_{\overline{M}}$  to the  $\kappa'$ -solenoid  $\sum_{\overline{M}'}$ . Then there is a unique map  $H : (\mathbb{R}^\kappa, \mathbf{0}) \rightarrow (\mathbb{R}^{\kappa'}, \mathbf{0})$  making the following diagrams commute

$$\begin{array}{ccc} & (\mathbb{R}^{\kappa'}, \mathbf{0}) & \\ H \nearrow & \downarrow \pi_{\overline{M}'} & \\ (\mathbb{R}^\kappa, \mathbf{0}) & \xrightarrow{h \circ \pi_{\overline{M}}} (\sum_{\overline{M}'}, e_{\overline{M}'}) & \text{or} \quad \begin{array}{ccc} (\mathbb{R}^\kappa, \mathbf{0}) & \xrightarrow{H} & (\mathbb{R}^{\kappa'}, \mathbf{0}) \\ \pi_{\overline{M}} \downarrow & & \downarrow \pi_{\overline{M}'} \\ (\sum_{\overline{M}}, e_{\overline{M}}) & \xrightarrow{h} & (\sum_{\overline{M}'}, e_{\overline{M}'}) \end{array} \end{array}$$

[S; 2.4.2]. Then let  $x, y$  be any points of  $\mathbb{R}^\kappa$ . Then

$$\begin{aligned} \pi_{\overline{M}'}(H(x) + H(y)) &= \pi_{\overline{M}'} \circ H(x) + \pi_{\overline{M}'} \circ H(y) = h \circ \pi_{\overline{M}}(x) + h \circ \pi_{\overline{M}}(y) \\ &= h \circ \pi_{\overline{M}}(x + y) = \pi_{\overline{M}'}(H(x + y)). \end{aligned}$$

From this it follows that  $H(x) + H(y) - H(x + y) \in \ker \pi_{\overline{M}'} \subset \mathbb{Z}^{\kappa'}$ . Define the following map

$$\lambda : \mathbb{R}^\kappa \times \mathbb{R}^\kappa \rightarrow \mathbb{R}^{\kappa'}; (x, y) \mapsto H(x) + H(y) - H(x + y).$$

By the above,  $\lambda(\mathbb{R}^\kappa \times \mathbb{R}^\kappa)$  is a connected subset of the totally disconnected set  $\mathbb{Z}^{\kappa'}$  and  $\lambda((\mathbf{0}, \mathbf{0})) = \mathbf{0}$ . Thus,  $\lambda(\mathbb{R}^\kappa \times \mathbb{R}^\kappa) = \mathbf{0}$  and  $H(x) + H(y) = H(x + y)$  for all  $(x, y) \in \mathbb{R}^\kappa \times \mathbb{R}^\kappa$  and  $H$  is a homomorphism.

Suppose then that  $h, h' \in Hom(\sum_{\overline{M}}, \sum_{\overline{M}'})$  and that  $h \neq h'$ . Since, for irrational  $\omega$ ,  $\Lambda_{\overline{M}}^\omega \subset \mathbf{C}_{\overline{M}}$  is dense we have that  $\mathbf{C}_{\overline{M}}$  is dense. Therefore there is an element  $\pi_{\overline{M}}(\mathbf{t}) \in \mathbf{C}_{\overline{M}}$  such that  $h(\pi_{\overline{M}}(\mathbf{t})) \neq h'(\pi_{\overline{M}}(\mathbf{t}))$  and hence  $\pi_{\overline{M}'}(H(\mathbf{t})) \neq \pi_{\overline{M}'}(H'(\mathbf{t}))$ , where  $H = f(h)$  and  $H' = f(h')$ . Then we must have  $H(\mathbf{t}) \neq H'(\mathbf{t})$ , and so  $f$  is one-to-one.

Suppose then that  $h \in Aut(\sum_{\overline{M}})$  and  $f(h) = H : \mathbb{R}^\kappa \rightarrow \mathbb{R}^{\kappa'}$  as above. Then we have  $h^{-1} \in Aut(\sum_{\overline{M}})$  and the corresponding endomorphism  $f(h') = H' : \mathbb{R}^\kappa \rightarrow \mathbb{R}^{\kappa'}$ . We then have that  $H' \circ H$  is the unique lifting of  $id \circ \pi_{\overline{M}}$  and  $id_{\mathbb{R}^{\kappa'}}$  also provides such a lifting, so we must have  $H' \circ H = id_{\mathbb{R}^{\kappa'}}$ . Similarly,  $H \circ H' = id_{\mathbb{R}^{\kappa'}}$  and  $H \in Aut(\mathbb{R}^{\kappa'})$ .  $\square$

**Lemma 3.10.** *If  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  is multiplication by the positive number  $a$  and if  $h \in Isomorphism(\sum_{\overline{M}}, \sum_{\overline{M}'})$  and if  $H : \mathbb{R}^\kappa \rightarrow \mathbb{R}^{\kappa'}$  satisfies  $h \circ \pi_{\overline{M}} = \pi_{\overline{M}'} \circ H$ , then*

$$\alpha \times h : \Phi_{\overline{M}}^\omega \overset{equiv}{\approx} \Phi_{\overline{M}'}^{\omega'}$$

where  $\omega' = \frac{1}{a}H(\omega) \in \mathbb{R}^{\kappa'}$ .

**Proof:** With  $\omega' = \frac{1}{a}H(\omega)$  we have:

$$\begin{aligned} h \circ \Phi_{\overline{M}}^\omega(t, x) &= h(\pi_{\overline{M}}(t\omega) + x) = h(\pi_{\overline{M}}(t\omega)) + h(x) = \pi_{\overline{M}'}(H(t\omega)) + h(x) \\ &= \pi_{\overline{M}'}(at\omega') + h(x) = \Phi_{\overline{M}'}^{\omega'}(at, h(x)) = \Phi_{\overline{M}'}^{\omega'} \circ (\alpha \times h)(t, x) \end{aligned}$$

$\square$



**Lemma 3.11.** *If the subsets  $\Lambda_M^\omega$  and  $\Lambda_M^{\omega'}$  of the  $\kappa$ -solenoid  $\Sigma_M$  are equal, then  $\omega = a\omega'$  for some  $a \in \mathbb{R} - \{0\}$ . And when  $\omega = a\omega'$ , with  $S \stackrel{def}{=} id_{\Sigma_M}$  if  $a > 0$  and  $S \stackrel{def}{=} \text{the map } x \mapsto -x$  if  $a < 0$ , we have*

$$(\alpha \times S) : \Phi_M^\omega \stackrel{equiv}{\approx} \Phi_M^{\omega'}, \text{ where } \alpha \text{ is multiplication by } |a|.$$

**Proof:** Suppose  $\Lambda_M^\omega = \Lambda_M^{\omega'} \stackrel{def}{=} \Lambda$  and let  $L^\omega = i^\omega(\mathbb{R}) \subset \mathbb{R}^\kappa$  and  $L^{\omega'} = i^{\omega'}(\mathbb{R}) \subset \mathbb{R}^\kappa$ . Since  $\Lambda$  is not a single point and any non-degenerate orbit is a locally one-to-one map from  $\mathbb{R}$ , there is an arc  $J = [0, x] \subset \mathbb{R}$  so that  $p \stackrel{def}{=} \pi_M \circ i^\omega|_J$  maps  $J$  homeomorphically onto its image in  $\Sigma_M$ . Then there is also an interval  $J' \subset \mathbb{R}$  having 0 as an endpoint such that  $p' \stackrel{def}{=} \pi_M \circ i^{\omega'}|_{J'}$  maps  $J'$  homeomorphically onto  $p(J) = p'(J')$ , and so there is a homeomorphism  $h : (J, 0) \rightarrow (J', 0)$  such that  $p' \circ h = p$ . Then

$$\begin{array}{ccc} (J', 0) & \xrightarrow{i^{\omega'}} & (\mathbb{R}^\kappa, \mathbf{0}) \\ h \uparrow & \searrow p' & \downarrow \pi_M \\ (J, 0) & \xrightarrow{p} & (\Sigma_M, e_M) \end{array},$$

and  $i^{\omega'} \circ h$  is the unique lift  $(J, 0) \rightarrow (\mathbb{R}^\kappa, \mathbf{0})$  of  $p$ . But  $i^\omega|_J$  also provides such a lift, and so  $i^{\omega'} \circ h = i^\omega|_J$  and  $i^\omega(J) \subset L^{\omega'}$ , from which it follows that  $L^\omega = L^{\omega'}$  and  $a\omega' = \omega$  for some  $a \in \mathbb{R}$  as claimed.

Suppose then that  $\omega = a\omega'$ . Then we have

$$\begin{aligned} S \circ \Phi_M^\omega(t, x) &= S(\pi_M(t\omega) + x) = S(\pi_M(t\omega)) + S(x) = \pi_M(|a|t\omega') + S(x) \\ &= \Phi_M^{\omega'}(|a|t, S(x)) = \Phi_M^{\omega'} \circ (\alpha \times S)(t, x) \end{aligned}$$

□

Thus, two linear flows  $\Phi_M^\omega$  and  $\Phi_M^{\omega'}$  whose trajectories determine the same decomposition of  $\Sigma_M$  (i.e., linear flows with the same phase portrait) are equivalent since then the trajectories of  $e_M$  ( $\Lambda_M^\omega$  and  $\Lambda_M^{\omega'}$  respectively) are equal.

**Definition 3.12.** *The homeomorphism  $h : X \rightarrow Y$  provides a topological equivalence between the flow  $\phi : \mathbb{R} \times X \rightarrow X$  and the flow  $\psi : \mathbb{R} \times Y \rightarrow Y$  if  $h$  maps each trajectory of  $\phi$  onto a trajectory of  $\psi$  and if  $h$  preserves the orientation of orbits; that is, for each  $x \in X$  there is an increasing homeomorphism  $\alpha_x : \mathbb{R} \rightarrow \mathbb{R}$  such that,  $h \circ \phi(t, x) = \psi(\alpha_x(t), h(x))$  for all  $t \in \mathbb{R}$  [I, p.32]. Such  $\phi$  and  $\psi$  are said to be topologically equivalent and we write*

$$h : \phi \stackrel{top}{\approx} \psi.$$

We proceed to determine the  $\stackrel{top}{\approx}$  classes of the families  $\mathcal{F}_M$ , and in the process we shall see that these  $\stackrel{top}{\approx}$  classes coincide with the  $\stackrel{equiv}{\approx}$  classes of these families. **Lemma 3.9** implies that we need only consider homeomorphisms  $h : (\Sigma_M, e_M) \rightarrow (\Sigma_M, e_M)$  since topological equivalence is more general than equivalence. We shall need a result from [Sch].

*Theorem 4.* Let  $G$  be a compact connected topological group, and let  $H$  be a locally compact abelian topological group. Then every  $f \in C_e(G, H) = \{\text{maps } G \rightarrow H \text{ mapping the identity of } G \text{ to the identity of } H\}$  is homotopic to exactly one

$h \in \text{Hom}(G, H)$ , and the homotopy can be chosen to preserve the identity ([Sch], **Corollary 2** of **Theorem 2**)

From this we immediately obtain the following corollary.

*Corollary 5.* Let  $G$  be a compact connected abelian topological group. Then any homeomorphism  $h : (G, e) \rightarrow (G, e)$  is homotopic to exactly one automorphism.

**Proof:** By our hypotheses on  $G$ , we obtain  $\alpha, \beta \in \text{Hom}(G, G)$  with  $\alpha$  homotopic to  $h$  and  $\beta$  homotopic to  $h^{-1}$ . Then  $\text{id}_G = h \circ h^{-1}$  is homotopic to  $\alpha \circ \beta \in \text{Hom}(G, G)$ . But by theorem, there is only one element of  $\text{Hom}(G, G)$  homotopic to  $h \circ h^{-1}$  and  $\text{id}_G = h \circ h^{-1} \in \text{Hom}(G, G)$ . Therefore,  $\alpha \circ \beta = \text{id}_G$  and similarly  $\beta \circ \alpha = \text{id}_G$ . Thus,  $\alpha$  is an automorphism whose uniqueness follows from the uniqueness in the above theorem.  $\square$

We can actually obtain the automorphism homotopic to  $h : (G, e) \rightarrow (G, e)$  as above in the following way. Start with the isomorphism  $h^*$  of the first Čech cohomology group of  $G$   $\check{H}^1(G)$  [ $\mathbb{Z}$  coefficients] induced by  $h$ . This then yields an automorphism  $\iota$  of the dual of  $G$   $\hat{G} \cong \check{H}^1(G)$ . The automorphism  $\hat{\iota}$ , the map of  $\hat{\hat{G}}$  dual to  $\iota$ , yields, via the automorphism  $G \cong \hat{\hat{G}}$  given by Pontryagin duality, an automorphism  $\alpha$  of  $G$ . This automorphism  $\alpha$  is the automorphism guaranteed by the above.

*Theorem 6.* If  $\Phi_M^\omega \stackrel{\text{top}}{\approx} \Phi_M^{\omega'}$ , there is an  $a \in \text{Aut}(\sum \overline{M})$  with  $(\beta \times a) : \Phi_M^\omega \stackrel{\text{equiv}}{\approx} \Phi_M^{\omega'}$ .

**Proof:** Suppose that  $h : (\sum \overline{M}, e_{\overline{M}}) \rightarrow (\sum \overline{M}, e_{\overline{M}})$  and  $h : \Phi_M^\omega \stackrel{\text{top}}{\approx} \Phi_M^{\omega'}$ . Then let  $a$  be the unique automorphism of  $\sum \overline{M}$  homotopic to  $h$  guaranteed by the above. Let  $F : (\sum \overline{M}, e_{\overline{M}}) \times [0, 1] \rightarrow (\sum \overline{M}, e_{\overline{M}})$  be a homotopy of  $h$  and  $a$  with  $F_t \stackrel{\text{def}}{=} F(-, t)$ . Then the map  $F' : (\sum \overline{M}, e_{\overline{M}}) \times [0, 1] \rightarrow (\sum \overline{M}, e_{\overline{M}}); (x, t) \mapsto F_t(x) - h(x)$  provides a homotopy between  $a - h$  and the constant map  $c : (\sum \overline{M}, e_{\overline{M}}) \rightarrow \{e_{\overline{M}}\}$ . Since  $\pi_{\overline{M}} : \mathbb{R}^\kappa \rightarrow \sum \overline{M}$  is a fibration and  $c$  is lifted by the constant map  $\sum \overline{M} \rightarrow \{\mathbf{0}\}$ , the map  $a - h$  can be lifted by a map  $g : (\sum \overline{M}, e_{\overline{M}}) \rightarrow (\mathbb{R}^\kappa, \mathbf{0})$  so that the following diagram commutes

$$\begin{array}{ccc} & & (\mathbb{R}^\kappa, \mathbf{0}) \\ & g \nearrow & \downarrow \pi_{\overline{M}} \\ (\sum \overline{M}, e_{\overline{M}}) & \xrightarrow{a-h} & (\sum \overline{M}, e_{\overline{M}}) \end{array}.$$

Let  $A \in \text{Aut}(\mathbb{R}^\kappa)$  be the map which satisfies  $\pi_{\overline{M}} \circ A = a \circ \pi_{\overline{M}}$ . Then with  $\varpi = A(\omega)$ , we have

$$a(\pi_{\overline{M}}(t\omega)) = \pi_{\overline{M}}(A(t\omega)) = \pi_{\overline{M}}(t\varpi),$$

and so  $a(\Lambda_M^\omega) = \Lambda_M^\varpi$ . If  $\Lambda_M^\varpi = \Lambda_M^{\omega'}$ , then **Lemma 3.11** gives us the desired result.

Suppose then that  $\Lambda_M^\varpi \neq \Lambda_M^{\omega'}$ . Now we have  $h : \Phi_M^\omega \stackrel{\text{top}}{\approx} \Phi_M^{\omega'}$  and so there is an increasing homeomorphism  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  such that,  $h(\pi_{\overline{M}}(t\omega)) = h \circ \Phi_M^\omega(t, e_{\overline{M}}) = \Phi_M^{\omega'}(\alpha(t), e_{\overline{M}}) = \pi_{\overline{M}}(\alpha(t)\omega')$  for all  $t \in \mathbb{R}$ , and since  $h(e_{\overline{M}}) = e_{\overline{M}}$  we have  $e_{\overline{M}} = \pi_{\overline{M}}(\alpha(0)\omega')$ . Now for  $n \in \mathbb{N}$  define the path

$$p_n : [0, n] \xrightarrow{i^\omega} \mathbb{R}^\kappa \xrightarrow{\pi_{\overline{M}}} \sum \overline{M} \xrightarrow{(a-h)} \sum \overline{M}.$$

Then  $\pi_{\overline{M}} \circ g \circ \pi_{\overline{M}} \circ i^\omega|_{[0,n]} = p_n$  and so  $g \circ \pi_{\overline{M}} \circ i^\omega|_{[0,n]}$  provides the unique lift  $([0,n], 0) \rightarrow (\mathbb{R}^\kappa, \mathbf{0})$  of the path  $p_n$ . But with

$$\gamma : ([0,n], 0) \rightarrow (\mathbb{R}^\kappa, \mathbf{0}); \quad t \mapsto t\varpi - \alpha(t)\omega' + \alpha(0)\omega',$$

we have

$$\begin{aligned} \pi_{\overline{M}} \circ \gamma(t) &= \pi_{\overline{M}}(t\varpi - \alpha(t)\omega' + \alpha(0)\omega') = \pi_{\overline{M}}(t\varpi) - \pi_{\overline{M}}(\alpha(t)\omega') \\ &= a(\pi_{\overline{M}}(t\omega)) - h(\pi_{\overline{M}}(t\omega)) = (a-h) \circ (\pi_{\overline{M}} \circ i^\omega)(t) = p_n(t). \end{aligned}$$

Thus, the uniqueness of  $g \circ \pi_{\overline{M}} \circ i^\omega|_{[0,n]}$  yields  $\gamma = g \circ \pi_{\overline{M}} \circ i^\omega|_{[0,n]}$  and  $g(\pi_{\overline{M}}(n\omega)) = n\varpi - \alpha(n)\omega' + \alpha(0)\omega'$ . Let  $\{\pi_{\overline{M}}(n_j\omega)\}_{j=1}^\infty \rightarrow x$  be a convergent subsequence of the sequence  $\{\pi_{\overline{M}}(n\omega)\}_{n=1}^\infty$ .

With  $L^\varpi = \{t\varpi \in \mathbb{R}^\kappa \mid t \in \mathbb{R}\}$  and  $L^{\omega'} = \{t\omega' \in \mathbb{R}^\kappa \mid t \in \mathbb{R}\}$ , we have  $L^\varpi \neq L^{\omega'}$  [if  $L^\varpi = L^{\omega'}$  there is an  $a \in \mathbb{R}$  such that  $a\omega' = \varpi$  and  $\Lambda_{\overline{M}}^\varpi = \Lambda_{\overline{M}}^{\omega'}$ ]. Then with  $L \stackrel{\text{def}}{=} L^\varpi \oplus L^{\omega'}$ , a closed two-dimensional vector subspace of  $\mathbb{R}^\kappa$ , we define the topological isomorphism  $\lambda : \mathbb{R}^2 \rightarrow L; (s, t) \mapsto s\omega + t\omega'$  (see, e.g., [Scha], Chapt1§3). Notice that

$$g(x) \in \overline{\{g \circ \pi_{\overline{M}}(n_j\omega)\}_{j=1}^\infty} = \overline{\{n_j\varpi - \alpha(n_j)\omega' + \alpha(0)\omega'\}_{j=1}^\infty} \subset L.$$

We then have in  $\mathbb{R}^2$

$$\begin{aligned} \lambda^{-1} \circ g(x) &= \lim_j \{\lambda^{-1} \circ g(\pi_{\overline{M}}(n_j\omega))\} \\ &= \lim_j \{\lambda^{-1}(n_j\varpi - \alpha(n_j)\omega' + \alpha(0)\omega')\} = \lim_j \{(n_j, -\alpha(n_j) + \alpha(0))\}, \end{aligned}$$

which is impossible since  $\{(n_j, -\alpha(n_j) + \alpha(0))\}$  is unbounded. We must therefore have  $\Lambda_{\overline{M}}^\varpi = \Lambda_{\overline{M}}^{\omega'}$ .  $\square$

This reduces the problem of determining the  $\overset{\text{equiv}}{\approx}$  and  $\overset{\text{top}}{\approx}$  classes of the families  $\mathcal{F}_{\overline{M}}$  to determining the image of  $f : \text{Aut}(\sum_{\overline{M}}) \rightarrow \text{Aut}(\mathbb{R}^\kappa)$  as in **Theorem 3**. Two linear flows  $\Phi_{\overline{M}}^\omega$  and  $\Phi_{\overline{M}}^{\omega'}$  are (topologically) equivalent if and only if there is an  $h \in \text{Aut}(\sum_{\overline{M}})$  whose lift  $H$  to an automorphism of  $\mathbb{R}^\kappa$  satisfies:  $a\omega' = H(\omega)$  for some  $a \in \mathbb{R} - \{0\}$ . Generally, if  $\Phi_{\overline{M}}^\omega \overset{\text{equiv}}{\approx} \Phi_{\overline{M}}^{\omega'}$  and the rank of the subgroup of  $(\mathbb{R}, +)$  generated by  $\{\omega_1, \omega_2, \dots\}$  is  $\rho$ , then the rank of the subgroup generated by  $\{\omega'_1, \omega'_2, \dots\}$  is also  $\rho$ : the closure of the corresponding trajectory in each case is a  $\rho$ -solenoid. However, it is important to realize that the image  $f(\text{Aut}(\sum_{\overline{M}}))$  depends on the  $\kappa$ -solenoid  $\sum_{\overline{M}}$  and so which flows of the same rank are equivalent depends on  $\sum_{\overline{M}}$ . Next we shall give a specific classification of these automorphisms on  $n$ -solenoids whose bonding maps are all represented by diagonal matrices, which correspond to the finite product of 1-solenoids.

#### 4. CLASSIFYING AUTOMORPHISMS ON THE FINITE PRODUCT OF 1-SOLENOIDS

**4.1. Comparing 1-Solenoids.** For a sequence of non-zero integers  $P = (p_1, p_2, \dots)$ , we have the corresponding 1-solenoid  $\sum_P$  where the bonding map  $f_i^{i+1}$  is multiplication by  $p_i$  in  $\mathbf{T}^1$ . [This is consistent with our established terminology; usually the  $p_i$  are required to be positive primes, but we include the case  $p_i = 1$ ].

**Definition 4.1.** Sequences of non-zero integers  $P = (p_1, p_2, \dots)$  and  $Q = (q_1, q_2, \dots)$  are equivalent, denoted  $P \cong Q$ , if and only if  $\sum_P$  is topologically isomorphic with  $\sum_Q$ . We let  $\bar{P}$  denote the  $\cong$  class of  $P$ .

**Definition 4.2.** For a given sequence of non-zero integers  $P = (p_1, p_2, \dots)$ ,  $|P| \stackrel{\text{def}}{=} (|p_1|, |p_2|, \dots)$ .

**Proposition 4.3.**  $P \cong |P|$ .

**Proof:** With  $\text{sgn}(i)$  is defined as the map  $S^1 \rightarrow S^1$  given by multiplication by  $\text{sign}(p_1 \cdots p_i)$ , we have the following topological isomorphism  $\sum_P \rightarrow \sum_{|P|}$  represented by the vertical maps in the following commutative diagram

$$\begin{array}{ccccccc} S^1 & \xleftarrow{p_1} & S^1 & \xleftarrow{p_2} & S^1 & \xleftarrow{\quad} & \dots \\ id \downarrow & & \text{sgn}(1) \downarrow & & \text{sgn}(2) \downarrow & & \\ S^1 & \xleftarrow{|p_1|} & S^1 & \xleftarrow{|p_2|} & S^1 & \xleftarrow{\quad} & \dots \end{array}$$

□

**Definition 4.4.** Given a sequence  $P = (p_1, p_2, \dots)$  of non-zero integers, we define the derived sequence  $P' = (p'_1, p'_2, \dots)$  to be the equivalent sequence of primes and 1's obtained by the prime factorization of the  $|p_i|$  in sequence, with the factors of  $|p_i|$  ordered by magnitude [1's are left unchanged].

For example: for  $P = (6, 1, -90, \dots)$ ,  $P' = (2, 3, 1, 2, 3, 3, 5, \dots)$ .

**Definition 4.5.** We define the partial order  $\leq$  on sequences of non-zero integers:  $P \leq Q$  iff a finite number of terms may be deleted from  $P'$  so that each prime occurring in this deleted sequence occurs in  $Q'$  with the same or greater cardinality.

Since  $(1, 1, \dots)$  is a minimal element of the partial order which is equivalent with any other minimal element, we can summarize the classification of all 1-solenoids in this new terminology as follows. This classification was conjectured by Bing [B], while the first proof in print appears in [McC]; see also [AF].

*Theorem 7.*  $([P \leq Q] \text{ and } [Q \leq P]) \Leftrightarrow [P \cong Q]$ .

And so  $\leq$  induces a partial ordering  $\preceq$  on  $\cong$  classes.

**Definition 4.6.** We define the sequence of primes

$$\prod = (2, 3, 2, 5, 3, 2, \dots) = (p_1^1, p_1^2, p_2^1, p_1^3, p_2^2, p_3^1, \dots) = (\phi_1, \phi_2, \dots),$$

where  $p_i^k$  is the  $k^{\text{th}}$  prime for all  $i$ . That is, with  $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  denoting the standard “diagonal” bijection given by  $(k, i) \mapsto \frac{(k+i-2)(k+i-1)}{2} + i$ ,  $\prod$  is the sequence of primes whose  $j^{\text{th}}$  member  $\phi_j$  satisfies the following:  $\phi_j = p_i^k$ , where  $(k, i) = f^{-1}(j)$ ; equivalently,  $\phi_{f(k,i)} = p_i^k$ .

Notice that under the partial ordering  $\preceq$   $\overline{(1, 1, \dots)}$  is the minimum element and  $\overline{\prod}$  is the maximum element.

**Definition 4.7.** For the prime  $p$  and a sequence of primes and 1's  $P$ ,  $\text{card}_p(P)$  is the cardinality with which  $p$  occurs in  $P$ .

We now introduce a way of arranging finitely many sequences of integers which is convenient for our purposes.

**Definition 4.8.** Given sequences of non-zero integers

$$P_1 = (p_1^1, p_2^1, \dots), \dots, P_n = (p_1^n, p_2^n, \dots),$$

we define their proper arrangement to be the sequences

$$Q_1 = (q_1^1, q_2^1, \dots), \dots, Q_n = (q_1^n, q_2^n, \dots)$$

defined as follows: let  $\kappa \stackrel{\text{def}}{=} \max\{\text{card}_2(P'_1), \dots, \text{card}_2(P'_n)\}$  and  $m \stackrel{\text{def}}{=} \min\{i : \text{card}_2(P'_i) = \kappa\}$ . For all  $i \in \mathbb{N}$

$$q_{f(1,i)}^m \stackrel{\text{def}}{=} \begin{cases} 2 & \text{for } i \leq \kappa \\ 1 & \text{otherwise} \end{cases}, \begin{pmatrix} q_{f(1,i)}^m = 2 \text{ for all } i \text{ if } \kappa = \infty \\ q_{f(1,i)}^m = 1 \text{ for all } i \text{ if } \kappa = 0 \end{pmatrix}.$$

For any  $P_j$  satisfying  $P_m \leq P_j$  define  $q_{f(1,i)}^j = q_{f(1,i)}^m$  for all  $i \in \mathbb{N}$ . Let  $\{j_1, \dots, j_k\} \subset \{1, \dots, n\}$  be the indices  $\ell$  for which the terms  $q_{f(1,i)}^\ell$  have not been defined (possibly empty). Repeat the above procedure on  $P_{j_1}, \dots, P_{j_k}$ . Repeat the procedure as many times as needed until  $q_{f(1,i)}^j$  is defined for all  $j \in \{1, \dots, n\}$  and all  $i \in \mathbb{N}$ . Proceed recursively, the occurrence of the  $k^{\text{th}}$  prime in the sequences determines the values of  $q_{f(k,i)}^j$  for  $(j, i) \in \{1, \dots, n\} \times \mathbb{N}$ .

Notice that  $Q_i \cong P_i$  since to compose  $Q_i$  we add at most a finite number of prime factors to those of  $P'_i$ .

**Definition 4.9.**  $P_1 = (p_1^1, p_2^1, \dots), \dots, P_n = (p_1^n, p_2^n, \dots)$  are properly arranged iff the proper arrangement of  $P_1, \dots, P_n$  is  $P_1, \dots, P_n$ .

**Definition 4.10.** If  $P$  and  $Q$  are properly arranged and  $P \geq Q$  or if  $P = Q$ , for  $n \in \mathbb{Z}$  we define the map  $n_{P \rightarrow Q} : \sum_P \rightarrow \sum_Q$  to be the map represented by the vertical maps in the following commutative diagram

$$\begin{array}{ccccccc} S^1 & \xleftarrow{p_1} & S^1 & \xleftarrow{p_2} & S^1 & \xleftarrow{p_3} & \dots \\ n \downarrow & & nr_1 \downarrow & & nr_2 \downarrow & & \\ S^1 & \xleftarrow{q_1} & S^1 & \xleftarrow{q_2} & S^1 & \xleftarrow{q_3} & \dots \end{array},$$

where  $r_i = \frac{p_1 \cdots p_i}{q_1 \cdots q_i} \in \mathbb{Z} - \{0\}$  since  $P \geq Q$ . We also denote  $n_{P \rightarrow P}$  by  $n_P$ .

Then for  $n \neq 0$   $n_{P \rightarrow Q}$  is a topological epimorphism since the maps  $n, nr_1, \dots : S^1 \rightarrow S^1$  are topological epimorphisms.

#### 4.2. Solenoidal Arithmetic.

**Definition 4.11.** For the prime  $p$ , we say  $p \mid P$  ( $p$  divides  $P$ ) if and only if  $p$  divides infinitely many  $p_i \in P$ .

The following lemma is a translation of the result [K, 2.4] into our terminology; it may also be shown number theoretically.

**Lemma 4.12.** For the prime  $p \in \mathbb{N}$ ,  $p_P$  is a topological isomorphism  $\Leftrightarrow p \mid P$ .  $\square$

**Definition 4.13.** For  $r \in \mathbb{R}$ , define  $r^{P \rightarrow Q} : \mathbf{C}_P \rightarrow \mathbf{C}_Q$  by  $r^{P \rightarrow Q}(\pi_P(s)) = \pi_Q(rs)$  if  $P$  is not eventually 1. And if  $P$  is eventually 1 and the product of all the terms ( $\neq 1$ ) of  $P$  is  $p$ , then  $\pi_P : [0, p) \rightarrow \mathbf{C}_P$  is one-to-one and onto, and we define  $r^{P \rightarrow Q} : \mathbf{C}_P \rightarrow \mathbf{C}_Q$  by  $r^{P \rightarrow Q}(\pi_P(s)) = \pi_Q(rs)$ ,  $s \in [0, p)$ .

Notice that if  $P$  is eventually 1 and  $Q$  is not eventually 1,  $r^{P \rightarrow Q}$  is not continuous; for if it were, then  $\lim_{s \rightarrow p^-} r^{P \rightarrow Q}(\pi_P(s))$  would be  $\pi_Q(rp) \neq e_Q$ , while  $\lim_{s \rightarrow 0+} r^{P \rightarrow Q}(\pi_P(s)) = e_Q$  and  $\lim_{s \rightarrow p^-} \pi_P(s) = \lim_{s \rightarrow 0+} \pi_P(s) = e_P$ .

**Lemma 4.14.** *If  $r \in \mathbb{R} - \mathbb{Q}$ , then  $r^{P \rightarrow Q}$  is not continuous.*

**Proof:** If  $P$  is eventually 1, then  $r^{P \rightarrow Q}$  followed by projection onto the first  $S^1$  factor is multiplication by an irrational on  $S^1$ , which is not continuous. Suppose then that  $P$  is not eventually 1. Now  $e_Q \in J \stackrel{\text{def}}{=} \cup_{i \in \mathbb{Z}} \{\pi_Q((-\frac{1}{4}, \frac{1}{4}) + iq_1)\} = \cup_{i \in \mathbb{Z}} J_i$  is a basic open subset of  $\mathbf{C}_Q$ , and if  $r^{P \rightarrow Q}$  were continuous, there would be a basic open subset  $I = \cup_{i \in \mathbb{Z}} \{\pi_P((-\delta, \delta) + ip_1 \cdots p_k)\}$  containing  $e_P$  such that  $r^{P \rightarrow Q}(I) = \cup_{i \in \mathbb{Z}} \{\pi_Q((-\alpha, \alpha) + irp_1 \cdots p_k)\} = \cup_{i \in \mathbb{Z}} I_i \subset J$ , where  $\alpha = |r\delta|$ . But the centers of successive  $I_i$  have preimages under  $\pi_Q$  which are not spaced by an integral amount as the centers of the  $J_i$  are  $\Rightarrow$  there is some  $I_k$  which is not contained in any  $J_i$ , contradicting  $r^{P \rightarrow Q}(I) \subset J$ .  $\square$

**Lemma 4.15.** *If  $r = \frac{c}{d} \in \mathbb{Q} - \{0\}$  and  $d$  has a prime factor  $p$  which does not divide  $P$  and  $\gcd(c, d) = 1$ , then  $r^{P \rightarrow P}$  is not continuous.*

**Proof:** Let  $\{s_n\} = \{\pi_P(p_1 \cdots p_n)\}$ . Then  $\{s_n\} \rightarrow e_P$ . Since  $p$  does not divide  $P$ , there is an  $N$  such that for all  $n \geq N$ ,  $p$  does not divide  $p_n$ . For  $n \geq N$  the  $(N+1)^{\text{th}}$  coordinate of  $r^{P \rightarrow P}(s_n) = \frac{c p_N \cdots p_n}{d} = k + \frac{\ell}{d}$ , where  $k \in \mathbb{Z}$  and  $\ell \in \{1, \dots, d-1\} \Rightarrow d_1^\infty(r^{P \rightarrow P}(s_n), e_P) \geq \frac{1}{2^{N+1}d} \Rightarrow \{r^{P \rightarrow P}(s_n)\}$  does not converge to  $e_P$ .  $\square$

Notice that if  $r = \frac{1}{d}$  and all prime factors of  $d$  divide  $P$ , then  $r^{P \rightarrow P}$  is a topological isomorphism by **Lemma 4.12**.

**Lemma 4.16.** *If  $r = \frac{c}{d} \in \mathbb{Q} - \{0\}$  and  $P > Q$  and  $d$  has a prime factor  $p$  which does not divide  $P$ , then  $r^{P \rightarrow Q}$  is not continuous.*

**Proof:** Very similar to the above proof.  $\square$

**Lemma 4.17.** *If  $r = \frac{c}{d} \in \mathbb{Q}$  and  $P > Q$  are properly arranged and  $d$  is the product of primes which divide  $P$ , then  $r^{P \rightarrow Q}$  is continuous.*

**Proof:**  $r^{P \rightarrow Q} = c_{P \rightarrow Q} \circ (\frac{1}{d})^{P \rightarrow P}$   $\square$

**Lemma 4.18.** *If  $r = \frac{c}{d} \in \mathbb{Q} - \{0\}$  and if  $P < Q$  or  $P$  and  $Q$  are not comparable, then  $r^{P \rightarrow Q}$  is not continuous.*

**Proof:** Let  $\{s_n\} = \{\pi_P(p_1 \cdots p_n)\}$ . Then  $\{s_n\} \rightarrow e_P$ . By our hypothesis, there is an  $N$  such that  $q_1 \cdots q_N$  does not divide  $cp_1 \cdots p_m$  for all  $m$ . Then the  $(N+1)^{\text{th}}$  coordinate of  $r^{P \rightarrow Q}(s_n) = \frac{cp_1 \cdots p_n}{dq_1 \cdots q_N} = k + \frac{\ell}{dq_1 \cdots q_N}$ , where  $k \in \mathbb{Z}$  and  $\ell \in \{1, \dots, dq_1 \cdots q_N - 1\} \Rightarrow \{r^{P \rightarrow Q}(s_n)\}$  does not converge to  $e_Q$ .  $\square$

**Definition 4.19.** *We define  $r \in \mathbb{R}$  to be a proper  $P \rightarrow Q$  multiplier [or proper multiplier if the context is clear] if the corresponding function  $r^{P \rightarrow Q}$  is a topological homomorphism.*

Thus, we may summarize our above results as follows:

If  $P \geq Q$ , then  $r$  is a non-zero proper  $P \rightarrow Q$  multiplier if and only if  $r = \frac{c}{d} \in \mathbb{Q} - \{0\}$  and all prime factors of  $d$  divide  $P$ ; if  $P < Q$  or  $P$  and  $Q$  are not comparable, then there is no non-zero proper  $P \rightarrow Q$  multiplier.

**Definition 4.20.** When  $r$  is a proper  $P \rightarrow Q$  multiplier we define the corresponding map  $r_{P \rightarrow Q} : \sum_P \rightarrow \sum_Q$  to be the map which extends  $r^{P \rightarrow Q}$ .

Notice that  $r_{P \rightarrow Q}$  is well-defined since we can write it as the composition of topological isomorphisms  $\sum_P \rightarrow \sum_P$  (corresponding to the composition of maps given by the factors in the denominator) and the epimorphism  $n_{P \rightarrow Q}$  [when  $n \neq 0$ ] for some  $n \in \mathbb{Z}$ . Thus, we obtain the following result.

**Lemma 4.21.**  $r_{P \rightarrow Q}$  is a topological epimorphism for any non-zero proper multiplier  $r$ ; if  $P = Q$ , then  $r_{P \rightarrow P}$  is a topological isomorphism if and only if  $r$  and  $\frac{1}{r}$  are proper multipliers.  $\square$

**Definition 4.22.** We define  $r \in \mathbb{R}$  to be a  $(P)$  iso-multiplier if and only if  $r$  and  $\frac{1}{r}$  are proper  $P \rightarrow P$  multipliers.

#### 4.3. Cartesian Products of 1-Solenoids as $n$ -Solenoids and their Automorphisms.

**Definition 4.23.** For  $n$  sequences of non-zero integers  $P_1 = (p_1^1, p_2^1, \dots), \dots, P_n = (p_1^n, p_2^n, \dots)$  we define the  $n$ -solenoid  $\sum_{\overline{P}}$  to be the  $n$ -solenoid corresponding to the sequence of matrices

$$\overline{P} = \left( \begin{pmatrix} p_1^1 & & 0 \\ & \ddots & \\ 0 & & p_1^n \end{pmatrix}, \begin{pmatrix} p_2^1 & & 0 \\ & \ddots & \\ 0 & & p_2^n \end{pmatrix}, \dots \right).$$

**Definition 4.24.**  $d_{(P_1, \dots, P_n)}$  is the metric on the product  $\prod_{i=1}^n \sum_{P_i}$  given by

$$d_{(P_1, \dots, P_n)} \left( \left( \langle x_1^j \rangle_{j=1}^\infty, \dots, \langle x_n^j \rangle_{j=1}^\infty \right), \left( \langle y_1^j \rangle_{j=1}^\infty, \dots, \langle y_n^j \rangle_{j=1}^\infty \right) \right) \stackrel{\text{def}}{=} \sum_{i=1}^n \frac{1}{2^i} d_\infty \left( \langle x_i^j \rangle_{j=1}^\infty, \langle y_i^j \rangle_{j=1}^\infty \right).$$

**Lemma 4.25.**  $\prod_{i=1}^n \sum_{P_i}$  is topologically isomorphic to  $\sum_{\overline{P}}$ .

**Proof:** Define  $i : \prod_{i=1}^n \sum_{P_i} \rightarrow \sum_{\overline{P}}$  by

$$\left( \langle x_1^j \rangle_{j=1}^\infty, \dots, \langle x_n^j \rangle_{j=1}^\infty \right) \mapsto \left( \langle x_1^j \rangle_{j=1}^\infty, \dots, \langle x_n^j \rangle_{j=1}^\infty \right) \in \sum_{\overline{P}} \subset \prod_{j=1}^\infty \mathbf{T}^n.$$

The function  $i$  is well-defined since  $x_i^j = p_j^i x_i^{j+1}$ . By construction,  $i$  is an algebraic isomorphism. And  $i$  is a homeomorphism; in fact,  $i$  is an isometry:

$$\begin{aligned} d_{(P_1, \dots, P_n)} \left( \left( \langle x_1^j \rangle_{j=1}^\infty, \dots, \langle x_n^j \rangle_{j=1}^\infty \right), \left( \langle y_1^j \rangle_{j=1}^\infty, \dots, \langle y_n^j \rangle_{j=1}^\infty \right) \right) &= \sum_{i=1}^n \frac{1}{2^i} d_\infty \left( \langle x_i^j \rangle_{j=1}^\infty, \langle y_i^j \rangle_{j=1}^\infty \right) \\ &= \sum_{i=1}^n \frac{1}{2^i} \left( \sum_{j=1}^\infty \frac{1}{2^j} d_1 \left( \left( x_i^j, y_i^j \right) \right) \right) = \sum_{i=1}^n \sum_{j=1}^\infty \frac{1}{2^i 2^j} d_1 \left( x_i^j, y_i^j \right) \text{ and} \end{aligned}$$

$$\begin{aligned}
d_n^\infty \left( \mathbf{i} \left( \langle x_1^j \rangle_{j=1}^\infty, \dots, \langle x_n^j \rangle_{j=1}^\infty \right), \mathbf{i} \left( \langle y_1^j \rangle_{j=1}^\infty, \dots, \langle y_n^j \rangle_{j=1}^\infty \right) \right) &= d_n^\infty \left( \langle \mathbf{x}^j \rangle_{j=1}^\infty, \langle \mathbf{y}^j \rangle_{j=1}^\infty \right) = \\
\sum_{j=1}^\infty \frac{1}{2^j} d_n \left( \langle x_1^j, \dots, x_n^j \rangle, \langle y_1^j, \dots, y_n^j \rangle \right) &= \sum_{j=1}^\infty \frac{1}{2^j} \left( \sum_{i=1}^n \frac{1}{2^i} d_1 \left( \langle x_i^j, y_i^j \rangle \right) \right) \\
&= \sum_{j=1}^\infty \sum_{i=1}^n \frac{1}{2^j 2^i} d_1 \left( x_i^j, y_i^j \right).
\end{aligned}$$

□

Notice that on  $\prod_{i=1}^n \mathbf{C}_{P_i}$   $\mathbf{i}$  takes on the simple form

$$(\pi_{P_1}(t_1), \dots, \pi_{P_n}(t_n)) \mapsto \pi_{\overline{P}}((t_1, \dots, t_n)).$$

*Theorem 8.* If  $P_1 = (p_1^1, p_2^1, \dots), \dots, P_n = (p_1^n, p_2^n, \dots)$  are properly arranged and if  $h$  is an automorphism of  $\sum_{\overline{P}}$  with the corresponding automorphism  $H : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying  $h \circ \pi_{\overline{P}} = \pi_{\overline{P}} \circ H$  and if  $A = (a_{ij})$  is the matrix representing  $H$  and if  $A^{-1} = (b_{ij})$ , then the entries  $a_{ij}$  and  $b_{ij}$  are proper  $P_j \rightarrow P_i$  multipliers for each  $i, j \in \{1, \dots, n\}$ . And if  $A = (a_{ij})$  is an invertible matrix with inverse  $A^{-1} = (b_{ij})$  and if all the entries  $a_{ij}$  and  $b_{ij}$  are proper  $P_j \rightarrow P_i$  multipliers, then there is an automorphism of  $\sum_{\overline{P}}$  represented by  $A$ . In the notation of **Theorem 3**,

$$f \left( \text{Aut} \left( \sum_{\overline{P}} \right) \right) = \{A \in GL(n, \mathbb{Q}) : \text{the entries of } A \text{ and } A^{-1} \text{ are proper } P_j \rightarrow P_i \text{ multipliers}\}.$$

**Proof:** Let  $h, H$  and  $A$  be as in the statement of the theorem. Fix an entry  $a_{ij}$  of  $A$ . With  $\phi_k : \prod_{\ell=1}^n \sum_{P_\ell} \rightarrow \sum_{P_k}$  projection onto the  $k^{\text{th}}$  factor, we have that  $\phi_j \circ \mathbf{i}^{-1}$  maps  $R_j \stackrel{\text{def}}{=} \{\pi_{\overline{P}}(\mathbf{t}) \in \sum_{\overline{P}} \mid \mathbf{t} = (t_1, \dots, t_n) \in \mathbb{R}^n \text{ and } t_\ell = 0 \text{ for } \ell \neq j\}$  isomorphically onto  $\mathbf{C}_{P_j}$ :

$$\pi_{\overline{P}}((0, \dots, t, 0, \dots, 0)) \xrightarrow{\mathbf{i}^{-1}} (\pi_{P_1}(0), \dots, \pi_{P_j}(t), \pi_{P_{j+1}}(0), \dots, \pi_{P_n}(0)) \xrightarrow{\phi_j} (\pi_{P_j}(t)).$$

Thus, we have the map  $\mu \stackrel{\text{def}}{=} \phi_i \circ \mathbf{i}^{-1} \circ h \circ (\phi_j \circ \mathbf{i}^{-1})^{-1} |_{\mathbf{C}_{P_j}} : \mathbf{C}_{P_j} \rightarrow \mathbf{C}_{P_i} \subset \sum_{P_i}$ . And for  $t \in \mathbb{R}$ :

$$\begin{aligned}
\mu(\pi_{P_j}(t)) &= \phi_i \circ \mathbf{i}^{-1} \circ h(\pi_{\overline{P}}((0, \dots, t, 0, \dots, 0))) = \phi_i \circ \mathbf{i}^{-1} \circ \pi_{\overline{P}} \circ H((0, \dots, t, 0, \dots, 0)) \\
&= \phi_i \circ \mathbf{i}^{-1} \circ \pi_{\overline{P}}(a_{1j}t, \dots, a_{nj}t) = \pi_{P_i}(a_{ij}t),
\end{aligned}$$

and so  $a_{ij}$  is a proper  $P_j \rightarrow P_i$  multiplier since the map  $\mu$  equals  $(a_{ij})^{P_j \rightarrow P_i}$ . Similarly, each  $b_{ij}$  is a proper  $P_j \rightarrow P_i$  multiplier since  $A^{-1}$  represents  $h^{-1}$ .

Given an invertible matrix  $A = (a_{ij})$  with inverse  $A^{-1} = (b_{ij})$  where the entries  $a_{ij}$  and  $b_{ij}$  are proper  $P_j \rightarrow P_i$  multipliers for all  $i, j \in \{1, \dots, n\}$ , we define the map  $A_{\overline{P}}$  to be the map on  $\sum_{\overline{P}}$  conjugate via  $\mathbf{i}^{-1}$  to the map  $\mathcal{A} : \prod_{\ell=1}^n \sum_{P_\ell} \rightarrow \prod_{\ell=1}^n \sum_{P_\ell}$ :

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \xrightarrow{\mathcal{A}} \begin{pmatrix} (a_{11})_{P_1 \rightarrow P_1}(x_1) & + \dots + & (a_{1n})_{P_n \rightarrow P_1}(x_n) \\ & \vdots & \\ (a_{n1})_{P_1 \rightarrow P_n}(x_1) & + \dots + & (a_{nn})_{P_n \rightarrow P_n}(x_n) \end{pmatrix}; A_{\overline{P}} \stackrel{\text{def}}{=} \mathbf{i} \circ \mathcal{A} \circ \mathbf{i}^{-1}.$$



Then  $A_{\overline{P}} \circ \pi_{\overline{P}} = \pi_{\overline{P}} \circ A$  as desired and  $(A_{\overline{P}})^{-1} = (A^{-1})_{\overline{P}}$ .  $\square$

Any finite product of 1-solenoids is isomorphic to the finite product of properly arranged 1-solenoids and the isomorphism will yield equivalences between the two corresponding families of linear flows [**Lemma 3.10**], so we need only consider the families of flows on the finite product of properly arranged 1-solenoids for the purposes of classification. And two linear flows  $\Phi_{\overline{P}}^{\omega}$  and  $\Phi_{\overline{P}}^{\omega'}$  on a product of properly arranged 1-solenoids are equivalent if and only if there is an automorphism  $A_{\overline{P}}$  with  $a\omega' = A(\omega)$  for some  $a \in \mathbb{R} - \{0\}$  by the above and by **Theorem 6**.

**4.4. Example: Classification in Dimension 2.** We only consider properly arranged  $P$  and  $Q$ . We classify the linear flows on  $\sum_{(P,Q)} \stackrel{def}{=} i(\sum_P \times \sum_Q)$ . There are then 3 cases:

1.  $P = Q$
2.  $P > Q$  [i.e.,  $P \geq Q$  but not  $Q \geq P$ ]
3.  $P$  and  $Q$  are not comparable.

*Corollary 9.* All equivalences of linear flows on  $\sum_{(P,P)}$  are generated by automorphisms of the form  $A_{P \times P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}_{P \times P}$  for which all entries of  $A_{P \times P}$  and  $A_{P \times P}^{-1}$  are proper  $P \rightarrow P$  multipliers. In particular, all rational linear flows on  $\sum_{(P,P)}$  are equivalent.

**Proof:** The first statement follows from **Theorem 8**. Choosing appropriate integers  $a, b, c$  and  $d$  we can equate any two rational flows with an automorphism of the form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{P \times P}$  where  $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \pm 1$  (see [I], p.36).  $\square$

The special case  $P = (1, 1, \dots)$  corresponds to the torus and the classical classification of linear flows on the torus: in this case the only proper  $P \rightarrow P$  multipliers are integers and  $f(Aut(\sum_{(P,P)})) = GL(2, \mathbb{Z})$ .

*Corollary 10.* If  $P > Q$  are properly arranged, then all equivalences of irrational flows on  $\sum_{(P,Q)}$  may be induced by automorphisms of the form  $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}_{(P,Q)}$ , where  $c$  and  $\frac{c}{ad}$  are proper  $P \rightarrow Q$  multipliers and  $a$  and  $d$  are iso-multipliers.  $\square$

*Corollary 11.* If  $P$  and  $Q$  are not comparable, then all equivalences of linear flows on  $\sum_{(P,Q)}$  may be induced by automorphisms of the form  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}_{(P,Q)}$ , where  $a$  and  $d$  are iso-multipliers.

**Proof:** By the hypothesis  $b$  and  $c$  must be 0, since this is the only proper  $Q \rightarrow P$  ( $P \rightarrow Q$ ) multiplier. Any such  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}_{(P,Q)}$  as stated is an automorphism with inverse  $\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}_{(P,Q)}$ .  $\square$

A somewhat surprising example of this case is given by  $P = (2, 5, \dots)$  and  $Q = (3, 7, \dots)$ , the sequences of the odd and even indexed primes. Then  $f(Aut(\sum_{(P,Q)})) = \left\{ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \right\}$  since neither  $P$  nor  $Q$  has a prime divisor, implying that at most

two distinct phase portraits of linear flows are represented by any given equivalence class of  $\mathcal{F}_{(P,Q)}$ .

In general, **Theorem 8** may be used to determine whether a given matrix represents an element of  $Aut\left(\sum_{(P_1, \dots, P_n)}\right)$ . When  $P_1 = \dots = P_n$  or when  $P_1, \dots, P_n$  are pairwise incomparable, the classification works out as in **Corollaries 9** and **11** respectively. However, there are many other possibilities besides these in the general case corresponding to the one case **10** as above when  $n = 2$ . For example, when  $n = 3$  there are 7 additional distinct possibilities. When  $P_1 = \dots = P_n = (1, 1, \dots)$ , we obtain the well known result  $f\left(Aut\left(\sum_{(P_1, \dots, P_n)}\right)\right) = GL(n, \mathbb{Z})$  since the only proper  $P_j \rightarrow P_i$  multipliers are integers, giving the classification of linear flows on  $\mathbf{T}^n$ .

## 5. APPENDIX

Now we find simple conditions on the character group of an  $n$ -solenoid that determine when it is a product of 1-solenoids. For examples of 2-solenoids which are not such products see [KM] and [GR]. Recall that a  $\mathbb{Q}$ -basis for a subgroup  $G \subset (\mathbb{R}, +)$  is a set  $B = \{\beta_i\}_{i \in I}$  satisfying the property that all  $g \in G - \{0\}$  can be uniquely represented as a sum  $g = \frac{c_1}{d_1}\beta_{i_1} + \dots + \frac{c_n}{d_n}\beta_{i_n}$ , where for  $i = 1, \dots, n$   $\frac{c_i}{d_i} \in \mathbb{Q} - \{0\}$  and  $\gcd(c_i, d_i) = 1$ . And we refer to the representation  $g = \sum_{i \in I} \frac{c_i}{d_i}\beta_i$  with  $\frac{c_i}{d_i} = \frac{0}{1}$  for all 0 terms as the canonical representation of  $g$ . We shall need the following definition.

**Definition 5.1.** We define a set of generators  $\mathcal{S}$  for a group  $G \subset (\mathbb{R}, +)$  to be relatively prime with respect to the  $\mathbb{Q}$ -basis  $\{\beta_1, \dots, \beta_n\}$  if for each  $\lambda \in \mathcal{S}$  with canonical representation

$$\lambda \stackrel{can}{=} \frac{c_1}{d_1}\beta_1 + \dots + \frac{c_n}{d_n}\beta_n$$

we have  $\gcd\left(d_j, \frac{d}{d_j}\right) = 1$  for  $j = 1, \dots, n$ , where  $d = d_1 \times \dots \times d_n$ .

**Theorem 12.** The  $n$ -solenoid  $\sum_{\overline{N}}$  is isomorphic with a product of  $n$  1-solenoids  $\Leftrightarrow$  its character group  $\widehat{\sum_{\overline{N}}}$  is isomorphic with a countable subgroup of the reals  $G$  (in the discrete topology) which has a set of generators  $\mathcal{S} = \left\{\lambda_i \stackrel{can}{=} \frac{c_1^i}{d_1^i}\beta_1 + \dots + \frac{c_n^i}{d_n^i}\beta_n\right\}$  which is relatively prime with respect to a  $\mathbb{Q}$ -basis  $B = \{\beta_1, \dots, \beta_n\} \subset G$ .

**Proof:**( $\Rightarrow$ ) Suppose the conditions of the theorem are met. Then we let

$$\delta_j^0 \stackrel{def}{=} d_j^0 \stackrel{def}{=} 1 \text{ (for } j = 1, \dots, n) \text{ and recursively}$$

$$\text{for } i \geq 1 \text{ and all } j = 1, \dots, n \text{ we let } \delta_j^i \stackrel{def}{=} \text{lcm}(d_j^i, \delta_j^{i-1}) \text{ and } \Delta_j^i \stackrel{def}{=} \frac{\delta_j^i}{\delta_j^{i-1}}$$

$$\text{and } M_i \stackrel{def}{=} \begin{pmatrix} \Delta_1^i & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \Delta_n^i \end{pmatrix} \text{ and } \overline{M} \stackrel{def}{=} (M_1, M_2, \dots)$$

$$\text{and } \mathcal{G} \stackrel{def}{=} \left\{\beta_1, \dots, \beta_n, \frac{\beta_1}{\delta_1^1}, \dots, \frac{\beta_n}{\delta_n^1}, \dots, \frac{\beta_1}{\delta_1^i}, \dots, \frac{\beta_n}{\delta_n^i}, \dots\right\}$$

$$\text{and } \mathfrak{M} \stackrel{def}{=} \text{the subgroup of } (\mathbb{R}, +) \text{ generated by } \mathcal{G}.$$

Since  $\mathfrak{M}$  is a subgroup of  $(\mathbb{R}, +)$  and

$$\frac{\delta_j^i}{d_j^i} \in \mathbb{Z}, \text{ we have } \frac{1}{d_j^i} \beta_j = \frac{\delta_j^i}{d_j^i} \frac{1}{\delta_j^i} \beta_j \in \mathfrak{M} \text{ for any } i, j.$$

And so we have  $\lambda_i \subset \mathfrak{M}$  for all  $i$ . Thus,  $G \subset \mathfrak{M}$  and to show  $G = \mathfrak{M}$  it suffices to show  $G \supset \mathcal{G}$ . Assume inductively on  $i \in \mathbb{N}$  that

$$\frac{1}{d_j^{i-1}} \beta_j \text{ and } \frac{1}{\delta_j^{i-1}} \beta_j \in G \text{ for } j = 1, \dots, n$$

(this clearly holds for  $i = 1$ ). Let  $d^i \stackrel{\text{def}}{=} d_1^i \cdots d_n^i$ . For  $k \neq j$ ,  $\frac{c_k^i d^i}{d_j^i d_k^i} \in \mathbb{Z} \Rightarrow$

$$\frac{c_j^i d^i / d_j^i}{d_j^i} \beta_j = \frac{c_j^i d^i}{d_j^i d_j^i} \beta_j = \frac{d^i}{d_j^i} \lambda_i - \frac{c_1^i d^i}{d_j^i d_1^i} \beta_1 - \cdots - \frac{c_{j-1}^i d^i}{d_j^i d_{j-1}^i} \beta_{j-1} - \frac{c_{j+1}^i d^i}{d_j^i d_{j+1}^i} \beta_{j+1} - \cdots - \frac{c_n^i d^i}{d_j^i d_n^i} \beta_n \in G.$$

Since  $\mathcal{G}$  is relatively prime with respect to  $B$ ,  $\gcd(\frac{c_j^i d^i}{d_j^i}, d_j^i) = 1$  and so there are

integers  $\mu$  and  $\nu$  such that  $\mu \frac{c_j^i d^i}{d_j^i} + \nu d_j^i = 1 \Rightarrow$

$$\frac{1}{d_j^i} \beta_j = \frac{\mu c_j^i d^i / d_j^i + \nu d_j^i}{d_j^i} \beta_j = \mu \frac{c_j^i d^i / d_j^i}{d_j^i} \beta_j + \nu \beta_j \in G.$$

Then we also have  $\gcd(\frac{\delta_j^i}{d_j^i}, \frac{\delta_j^i}{\delta_j^{i-1}}) = 1$  and so there are integers  $r$  and  $s$  so that

$$r \frac{\delta_j^i}{d_j^i} + s \frac{\delta_j^i}{\delta_j^{i-1}} = 1. \text{ Hence,}$$

$$\frac{1}{\delta_j^i} \beta_j = \frac{r \frac{\delta_j^i}{d_j^i} + s \frac{\delta_j^i}{\delta_j^{i-1}}}{\delta_j^i} \beta_j = r \frac{1}{d_j^i} \beta_j + s \frac{1}{\delta_j^{i-1}} \beta_j \in G,$$

completing the inductive step. Thus,  $G = \mathfrak{M}$  and since  $\mathfrak{M}$  is isomorphic with the direct limit of

$$\mathbb{Z}^n \xrightarrow{M_1} \mathbb{Z}^n \xrightarrow{M_2} \mathbb{Z}^n \xrightarrow{M_3} \cdots,$$

we have by Pontryagin duality that  $\sum_{\overline{N}} \cong \sum_{\overline{M}}$ , which in turn is topologically isomorphic with a product of  $n$  1-solenoids [Lemma 4.25]. The other direction is clear since duality respects finite products.  $\square$

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